BC 6

$$f(x)=\sum_{n=2}^{\infty}\frac{(-1)^n(2x)^n}{n-1}$$

(a) By the ratio test, the series converges when $\lim_{n\to\infty}\left|\dfrac{\frac{(2x)^{n+1}}{n}}{\frac{(2x)^n}{n-1}}\right|<1.$ Evaluating the limit gives

$$\lim_{n\to\infty}\left|\frac{\frac{(2x)^{n+1}}{n}}{\frac{(2x)^n}{n-1}}\right|=\lim_{n\to\infty}\left(\frac{n}{n-1}\right)|2x|=|2x|<1 \text{ or } -\frac{1}{2}<x<\frac{1}{2}.$$

At $x=-\frac{1}{2}$, the series is $\sum_{n=2}^{\infty}\frac{(-1)^n(-1)^n}{n-1}=\sum_{n=2}^{\infty}\frac{1}{n-1}$. This is the harmonic series, which diverges. Or, using the integral test: $\int_2^{\infty}\frac{1}{x-1}dx=\lim_{b\to\infty}\left(\ln(x-1)\right)\Big|_2^b=\lim_{b\to\infty}\left(\ln(b-1)-\ln 1\right)=\infty$.
Hence the series diverges since the integral diverges.

At $x=\frac{1}{2}$, the series is $\sum_{2}^{\infty}\frac{(-1)^n}{n-1}$. This is an alternating series with terms decreasing by absolute value and approaching zero. Therefore, it converges by the Alternating Series Test.

The interval of convergence is $\boxed{-\frac{1}{2}<x\le\frac{1}{2}}$.

(b) $$y=\sum_{n=2}^{\infty}\frac{(-1)^n(2x)^n}{n-1}=\frac{(2x)^2}{1}-\frac{(2x)^3}{2}+\frac{(2x)^4}{3}-\frac{(2x)^5}{4}+\cdots=4x^2-4x^3+\frac{16}{3}x^4-8x^5+\cdots$$

$$y'=8x-12x^2+\frac{64}{3}x^3-40x^4+\cdots$$

$$xy'-y=\left(8x^2-12x^3+\frac{64}{3}x^4-40x^5+\cdots\right)-\left(4x^2-4x^3+\frac{16}{3}x^4-8x^5+\cdots\right)=$$

$$4x^2-8x^3+16x^4-32x^5+\cdots=\sum_{n=2}^{\infty}(-2x)^n.$$

This is a geometric series whose first term is $4x^2$ and common ratio $-2x$. Therefore, the sum of the series is equal to $\frac{4x^2}{1-(-2x)}=\frac{4x^2}{1+2x}$ when $|2x|<1$ or $-\frac{1}{2}<x<\frac{1}{2}$ which has the same radius of convergence form part (a).