A POSTAGE STAMP PROBLEM*

Ronald Alter and Jeffrey A. Barnett†

The postage stamp problem is the following: An envelope may carry no more than h stamps, and one has available k integer-valued stamp denominations. Given h and k, find the maximal integer $n = n(h,k)$ such that all integer postage values from 1 to n can be made up. In addition, find all sets of k stamp denominations satisfying this condition.

The problem statement is usually modified by augmenting the solution sets with a stamp of value zero, and requiring that a letter carry exactly h stamps. For example, if $h = 2$ and $k = 3$, then $n(h,k) = 8$. The unique solution set is $\{0, 1, 3, 4\}$. A construction of the integers $1, \ldots, 8$ is

$$\begin{array}{llll} 1 = 0+1 & 3 = 0+3 & 5 = 1+4 & 7 = 3+4 \\ 2 = 1+1 & 4 = 0+4 & 6 = 3+3 & 8 = 4+4. \end{array}$$

Many solution sets may exist. For example, $n(2,6) = 20$, and the five solution sets are $\{0,1,2,5,8,9,10\}$, $\{0,1,3,4,8,9,11\}$, $\{0,1,3,4,9,11,16\}$, $\{0,1,3,5,6,13,14\}$, and $\{0,1,3,5,7,9,10\}$.

Clearly, $n(1,k) = k$ with the solution set $\{0,1,\ldots,k\}$, and $n(h,1) = h$ with the solution set $\{0,1\}$. Stöhr [44], Henrici [12], and Stanton et al. [43], independently, show that

$$n(h,2) \;=\; \lfloor (h^2+6h+1)/4 \rfloor,$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. If h is odd, then the unique solution set is $\{0, 1, (h+3)/2\}$. If h is even, then there are two solution sets: $\{0, 1, (h+2)/2\}$ and $\{0, 1, (h+4)/2\}$.

The only other case of a near closed-form solution is $k = 3$. Hofmeister [17] shows that

$$\frac{4}{81}h^3 + \frac{2}{3}h^2 + \frac{66}{27}h \;\le\; n(h,3) \;\le\; \frac{4}{81}h^3 + \frac{2}{3}h^2 + \frac{71}{27}h - \frac{1}{81}, \quad h \ge 34$$

*This article appeared in *The American Mathematics Monthly* 87(3) pp. 206–210, 1980.
†Current email address: jbb@notatt.com.

The lower bound is achieved whenever $h \equiv 0 (\text{mod } 9)$. Klotz [20], [21] and Henrici [12] independently report similar but weaker results for the lower bound.

In 1936, Rohrbach [37] developed asymptotic bounds for n with h fixed and k large. He showed that

$$\left(\frac{k}{h}\right)^h \;\le\; n(h,k) \;\le\; \frac{k^h}{h!} + O(k^{h-1})$$

The lower bound is developed constructively. The upper bound is obtained by noting that $\binom{k+h}{h} - 1$ is sufficient because $\binom{k+h}{h}$ is the number of combinations of $k+1$ things (recall the zero stamp) taken h at a time with replacement. The -1 appears because 0 is a potential sum but illegal postage.

Hofmeister [19, p. 112] derives the best-known lower bound by using an unpublished result of R. Windecker:

$$n(h,k) \;\ge\; (4/3)^{\lfloor h/3 \rfloor}(8/7)^{\lfloor (h-3\lfloor h/3 \rfloor)/2 \rfloor}(k/h)^h - O(k^{h-1}).$$

Hofmeister [119, p. 104] also gives bounds for the case $k \ge 3$ fixed and h large, as

$$2^{\lfloor k/4 \rfloor}(4/3)^{\lfloor (k-4\lfloor k/4 \rfloor)/3 \rfloor}(h/k)^k + O(h^{k-1}) \;\le\; n(h,k) \;\le\; \frac{h^k}{k!} + O(h^{k-1}).$$

A nontrivial upper bound for $h = 2$ appears in Rohrbach [37] as

$$n(2,k) \;\le\; \frac{1}{2}(1 - 0.0016)k^2 + O(k).$$

This bound is improved in Klotz [20], [21], [22], who replaces 0.0016 by 0.0369.

Additional work on upper bounds for n is reported in Moser [27], Riddell [36], Salié [38], and Moser and Riddell [28]. For the case of large k, the best results to date are given by Moser et al. [29] as

$$n(h,k) \;<\; (1 - b_n)\frac{k^h}{h!},$$

where $b3 = 0.0221$ and $b4 = 0.0115$. Further, $b_h = (1.02f(h))^h$ when $h \ge 5$ and $b_h = (1.1f(h))^h$ when $h \ge 8$, where $f(h) = \cos(\pi/h)/(2 + \cos(\pi/h))$.

The American Mathematics Monthly 87 (3) 206–210, 1980.

Richard K. Guy suggests that, for h large enough, $n(h,k)$ is given by a finite set of polynomials in h of degree k. For example, Stöhr's solution for $k=2$ may be written $n(h,2)=(h^2+(3+3c)h+d)/4$, where $c=d=h,\ \bmod 2$. Guy's conjecture for $k=3$ is that, for $h \geq 20$,

$$n(h,3) \quad = \quad (4h^3+54h^2+(204+3c_r)h+d_r)/81,$$

where c_r, d_r are given, for $h \equiv r$, mod 9, by:

$r=$	-4	-3	-2	-1	0	1	2	3	4
$c_r=$	0	1	3	0	-2	0	3	1	0
$d_r=$	46	-81	-1	-170	0	62	-26	0	-154

This problem has been around for a long time; however, the earliest reference we could find is Rohrbach [37]. Several special cases of the postage stamp problem appear in the recreational literature. See, for example, Sprague [42, Prob. 18], Gardner [3, Prob. 4], and Legard [24].

A problem closely related to solving $n(2,k)$ is the representation of the integers $1,\ldots,n$ by differences of the members of a solution set. Miller [26] and Leech [23] describe this problem.

Alter and Barnett [1] describe an application of the $n(2,k)$ problem to the optimal allocation of index registers on computers. Hargraves [6] describes another application. He uses solution sets for $n(h,k)$ to design optimal wiring patterns for an associative cache memory.

Several thousand hours of computer time have been dedicated to obtaining values of $n(h,k)$. All reported algorithms are exponential in h and k. Table 1 summarizes the known values of n except those given by simple expressions (i.e., $h=1$ or $k=1,2$). The first publications of the included values are found in Stöhr [44], Henrici [12], Lunnon [25], Seldon [39], [40], Phillips [35], and Alter and Barnett [1]. Later, confirmatory results are given by Stanton et al. [43] and Heimer and Langenbach [11]. Henrici reports additional values for $n(2,k)$ where $k=14,\ldots,18$, as, respectively, 80, 92, 104, 116, and 128. He obtains these values using an unproved pruning heuristic. Thus they should be viewed as lower bounds until more reliable methods are employed. The values for $n(3,k)$, $k \leq 47$, were calculated by John A. Bate; we are grateful to him for allowing us to publish them.

Several special-case investigations are worthy of note. Wegner and Doig [45] examine symmetric denomination sets. Let $\nu=\{a_0=0<a_1<\cdots<a_k\}$ be a denomination set. Then ν is symmetric if the sequence of differences

Table 1
Known Values of $n(h, k)$

$h =$	$k =$ 3	4	5	6	7	8	9	10	11	12	13
2	8	12	16	20	26	32	40	46	54	64	72
3	15	24	36	52	70	93	121	154			
4	26	44	70	108	162	220					
5	35	71	126	211							
6	52	114	216	388							
7	69	165	345								
8	89	234	512				$= n(h, k)$				
9	112	326	797								
10	146	427									
11	172	547									
12	212	708									
13	259	873									
14	302	1094									

$k =$	15	16	17	18	19	20	21	22	23	24	25
$n(3, k) =$	354	418	476	548	633	714	805	902	1012	1127	1254
$k =$	26	27	28	29	30	31	32	33	34	35	36
$n(3, k) =$	1382	1524	1678	1841	2010	2188	2382	2584	2801	3020	3256
$k =$	37	38	39	40	41	42	43	44	45	46	47
$n(3, k) =$	3508	3772	4043	4326	4628	4941	5272	5606	5960	6334	6723

of consecutive elements is palindromic. Symmetric solution sets exist for all values of $n(2, k)$ that are known except $k = 10$. Rohrbach [37] investigates a restricted class of symmetric sets to derive his asymptotic bounds.

Henrici [12] drops the restriction that a_i be positive. He finds the solution set $\{-1, 2, 3, 4, 10, 11, 12, 15\}$ for $n(2, 7)$ and claims $n = 27$. Table 1 shows $n(2, 7) = 26$. Note, $k = 7$ by Henrici's definition. The claim is justified because the normal problem statement allows an uncounted zero element. On the other hand, this result has the range $1, \ldots, 27$, whereas the normal result has the range $0, \ldots, 26$.

Henrici finds the *symmetric* (and unique) solution set $\{-1, 1, 2, 4, 8, 12, 16, 20, 22, 23, 25\}$ with range $0, \ldots, 48$ for $n(2, 10)$. The value -1 is not included because sums must be formed from exactly two elements.

Alter and Barnett [1] derive an interesting bound for the case $h = k$. Namely, $n(h, h) \geq f_{2h} - 1$, where f_i is the ith Fibonacci number.

Since the initial statement of the postage stamp problem by Rohrbach, significant progress toward a solution has been made. However, many issues remain open.

The American Mathematics Monthly 87 (3) 206–210, 1980.

Problem 1. Can the bounds on n be improved? The distance between the best known upper and lower bounds is large. Clearly, there is room for progress short of finding a simple formula for n.

Problem 2. Is there a simple relation between $n(h,k)$ and $n(k,h)$?

Problem 3. What is the multiplicity of solution sets as a function of h and k?

Problem 4. Let $\nu = \{a_1,\dots,a_k\}$ and define $n(h,\nu)$ as the maximum integer, n, such that all integers $1,\dots,n$ can be made up as sums of no more than h of the a_i. Can $n(h,\nu)$ be expressed by a simple formula? Note that

$$n(h,k) = \max_{\nu \in U_k} n(h,\nu),$$

where U_k is the set of all k-element denomination sets.

Knowledge of $n(h,\nu)$ would be a tremendous aid to improving estimates of $n(h,k)$. The known lower bounds are generated by restricting U_k so that $n(h,\nu)$ is easily represented.

Problem 5. Let $\{a_1,\dots,a_k\}$ be a solution set for $n(h,k)$. What are bounds for a_i as a function of i, h, and k? Also, what is the magnitude of a_{i+1} relative to a_i?

Problem 6. What is the behavior of $n(h,k)$ if negative and rational stamp denominations are permitted?

Problem 7. For what values of h and k do symmetric solutions exist?

Problem 8. Do polynomial-time computational algorithms exist for $n(h,k)$ and the corresponding solution sets?

The bibliography includes several papers not cited in the text. Our literature search was more difficult than usual because the postage stamp problem seems to have been reinvented many times. Stöhr [44] summarizes work prior to 1955.

We would like to express our appreciation to M. L. V. Pitteway and R. K. Guy for helping us locate many of the articles cited here. Also, we thank N. Goldman for help with translation.

References

1. R. Alter and J. Barnett, Remarks on the postage stamp problem with applications to computers, Congressus Numerantium 19, Proc. Eighth Southeastern Conf. on Combinatorics, Graph Theory, and Comput., Baton Rouge, La., 1977, pp. 43–59.

2. M. Djawadi, Kennzeichnung von Mengen mit einer additiven Minimaleigenschaft, Diss. Joh. Guttenberg-Univ., Mainz, 1974.

3. M. Gardner, Mathematical games, A collection of short problems and more talk of prime numbers, Scientific American, 210 #6 (June 1964), Problem 4, p. 116; 211 #7 (July 1964) p. 114.

4. N. Hämmerer, Reichweite von Extremalbasen bei fester Ordnung, Diss. Joh. Guttenberg-Univ., Mainz, 1974.

5. N. Hämmerer and G. Hofmeister, Zu einer Vermutung von Rohrbach, J. Reine Angew. Math., 286/287 (1976) 239–247.

6. R. F. Hargraves, Jr., Application of the postage stamp problem to associative cache memory design, Dartmouth College, Hanover, N.H., 1971.

7. E. Härtter, Basen für Gitterpunktmengen, J. Reine Angew. Math., 202 (1959) 153–170.

8. ____, Einige Abschätzungen für Abschnittsbasen, J. Reine Angew. Math., 205 (1960/61) 82–90.

9. ____, Eine Bemerkung über Basen, Math. Ann., 165 (1966) 24–25.

10. ____, Additive Zahlentheorie, Vorlesung anider, Joh. Guttenberg-Univ., Mainz, Sommersemester 1973.

11. R. L. Heimer and H. Langenbach, The stamp problem, J. Rec. Math., 7 (1974) 235–250.

12. A. Henrici, The coins problem, Part 1, Diss., Diploma in Num. Anal. and Auto. Comput., Corpus Christi College, Cambridge, 1965.

13. G. Hofmeister, Methoden zur Abschätzung von $n(h,k)$ und $g(h,k)$ nach unten, Diplomarbeit, Freie Univ. Berlin, 1963.

14. ____, Über eine Menge von Abschnittsbasen, J. Reine Angew. Math., 213 (1963) 43–57.

15. ____, Zu einem Problem von Frobenius, Det Kgl. Norske Vidensk. Selsk. Skr. (1966) Nr. 5, 1–37.

16. ____, Über eine Menge von Abschnittsbasen 2, Det Kgl. Norske Vidensk. Selsk. Forhandlinger, 39 (1966) 60–65.

17. ____, Asymptotische Abschätzungen für dreielementige Extremalbasen in natürlichen Zahlen, J. Reint Angew. Math., 232 (1968) 77–101.

18. G. Hofmeister and H. Schell, Reichweiten von Mengen natürlicher Zahlen I, Det Kgl. Norske Vidensk Selsk. Skr. (1970) Nr. 10, 1–5.

19. G. Hofmeister, Endliche additive Zahlentheorie, Kapitel I, Das Reichweitenproblem, Joh. Guttenberg-Univ., Mainz, 1976.

20. W. Klotz, Extremalbasen mit fester Elementeanzahl, Diss. Tech. Univ. Carolo-Wilhelmina Braunschweig, 1968.

The American Mathematics Monthly 87 (3) 206–210, 1980.

21. ____, Extremalbasen mit fester Elementeanzahl, J. Reine Angew. Math., 237 (1969) 194–220.

22. ____, Eine obere Schranke für die Reichweite einer Extremalbasis zweiter Ordnung, J. Reine Angew. Math., 238 (1969) 161–168.

23. J. Leech, On the representation of $1, 2, \ldots, n$ by differences, J. London Math. Soc., 31 (1956) 160–169.

24. A. Legard, Brain-Teaser, Sunday Times, 23 Dec. 1962, and 20 Jan. 1963.

25. W. F. Lunnon, A postage stamp problem, Comput. J., 12 (1969) 377–380.

26. J. C. P. Miller, Difference bases: Three problems in additive number theory, in Symposium on Computers in Number Theory, Atkin and Birch, eds., Academic Press, 1971, pp. 299–322.

27. L. Moser, On the representation of $1, 2, \ldots, n$ by sums, Acta Arith., 6 (1960) 11–13.

28. L. Moser and J. Riddell, On additive h-bases for n, Colloq. Math., 9 (1962) 287–290.

29. L. Moser, J. R. Pounder, and J. Riddell, On the cardinality of h-bases for n, J. London Math. Soc., 44 (1969) 397–407.

30. A. Mrose, Die Bestimmung der extremalen regularen Abschnittsbasen mit der Hilfe einer Klasse von Kettenbruchdetenminanten, diss. Freie Univ., Berlin, 1969.

31. ____, Eine untere Schranke für die Reichweite von Extremalbasen dritter Ordnung, J. Reine Angew. Math., 261 (1973) 216–220.

32. ____, Ein rekursives Konstruktionsverfahren für Abschnittsbasen, J. Reine Angew. Math., 271 (1974) 214–217.

33. ____, Untere Schranken für Extremalbasen fester Ordnung, I, J. Reine Angew. Math. (im Druck).

34. H. H. Ostmann, Additive Zahlentheorie I, Berlin-Göttingen-Heidelberg, Springer, 1956.

35. B. P. Phillips, The postage stamp problem, Comput. J., 19 (1976) 93.

36. J. Riddell, On bases for sets of integers, thesis, Univ. of Alberta, 1960.

37. H. Rohrbach, Ein Beitrag zur additiven Zahlentheorie, Math. Z., 42 (1936) 1–30.

38. H. Salié, Reichweite von Mengen aus drei natürlichen Zahlen, Math. Ann., 165 (1966) 196–203.

The American Mathematics Monthly 87 (3) 206–210, 1980.

39. J. L. Seldon, The postage stamp problem, Comput. J., 15 (1972) 361.

40. ____, The ten stamp problem, Brunel Univ., Middlesex, 1973.

41. J. Smith, A note on the postage stamp problem (talk only), Combin. Math. III, Proc. Third Australian Conf., Brisbane, 1974.

42. R. P. Sprague, Recreations in Mathematics (trasl. by T. H. O'Beirne), Dover, New York, 1973.

43. R. G. Stanton, J. A. Bate, and R. C. Mullin, Some tables for the postage stamp problem, Proc. Fourth Manitoba Conf. on Numerical Math., 1974, pp. 351–356.

44. A. Stöhr, Gelöste und ungelöste Fragen über Basen der natürlichen Zahlenreihe I und II, J. Reine Angew. Math., 194 (1955) 40–65, 111–140.

45. P. Wegner and A. Doig, Symmetric solutions of the postage stamp problem, Rev. Franc. Recherche Op., 41 (1966) 353–374.

46. R. Windecker, Eine Abschnittsbasis dritter Ordnung, Det Kgl. Norske Vidensk. Selsk. Skr. (1976) Nr. 9, 1–3.

47. J. Zöllner, Über Mengen natürlicher Zahlen, für die jede euklidische Darstellung eine minimale Koeffizientensumme besitzt, Diplomarbeit, Joh. Guttenberg-Univ., Mainz, 1974.

Computer Science Department, University Of Kentucky, Lexington, KY 40506.

Information Sciences Institute, University Of Southern California, Marina Del Rey, CA 90291.

The American Mathematics Monthly 87 (3) 206–210, 1980.